change its length given by (1) if its endpoints are displaced according to (2). This is expressed by the equation:

$$\gamma_{\mu\nu,\sigma} A^\sigma + \gamma_{\mu\sigma} A^\sigma_{,\nu} + \gamma_{\nu\sigma} A^\sigma_{,\mu} = 0. \tag{3a}$$

The invariant character of this equation can easily be shown. Putting

$$\gamma_{\mu\sigma} A^\sigma = A_\mu, \tag{3b}$$

we get

$$A_{\mu,\nu} + A_{\nu,\mu} - A^\sigma\{\gamma_{\mu\sigma,\nu} + \gamma_{\nu\sigma,\mu} - \gamma_{\mu\nu,\sigma}\} = 0, \tag{3c}$$

or

$$\{A_{\mu,\nu} - A_\sigma \Gamma^\sigma_{\mu\nu}\} + \{A_{\nu,\mu} - A_\sigma \Gamma^\sigma_{\nu\mu}\} = 0, \tag{3d}$$

where $\Gamma^\lambda_{\iota\kappa}$ are the Christoffel symbols of the second kind. Or, in the language of the absolute tensor calculus we have:

$$A_{\mu;\nu} + A_{\nu;\mu} = 0.\text{[2]} \tag{3}$$

It follows from (3) that $\gamma_{\mu\nu} A^\mu A^\nu$ (that is the norm of A^μ) is constant along the lines to which A^μ is a tangent. By multiplying (3) by $A^\mu A^\nu$ we obtain:

$$A^\nu A^\mu A_{\mu;\nu} + A^\mu A^\nu A_{\nu;\mu} = A^\nu (A^\mu A_\mu)_{,\nu} = 0. \tag{4}$$

3. Kaluza's theory imposes another property on the vector field A^μ, besides the one expressed in (3): The lines to which the A^μ are tangents—the "A-lines"—have to be geodesics. Analytically this means: The lines defined by

$$\frac{dx^\nu}{d\sigma} = \lambda A^\nu \tag{5}$$

(where $1/\lambda^2$ is equal to the norm of A) satisfy the equation

$$\frac{d^2 x^\nu}{d\sigma^2} + \Gamma^\nu_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} = 0, \tag{6a}$$

or, according to (5):

$$\frac{\partial \ln \lambda}{\partial x^\alpha} A^\alpha A^\nu + A^\alpha A^\nu_{;\alpha} = 0. \tag{6b}$$

The first term vanishes because of (4) $\left(\frac{\partial \ln \lambda}{\partial x^\alpha} \cdot A^\alpha\right.$ is the derivative of a function of the norm of A in the direction of $\left.A\right)$. Therefore:

$$A^\alpha A^\nu_{;\alpha} = 0, \tag{6c}$$

[2] Killing's equation.

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ON A GENERALIZATION OF KALUZA'S THEORY OF ELECTRICITY

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INTRODUCTION

So far, two fairly simple and natural attempts to connect gravitation and electricity by a unitary field theory have been made, one by Weyl, the other by Kaluza. Furthermore, there have been some attempts to represent Kaluza's theory formally so as to avoid the introduction of the fifth dimension of the physical continuum. The theory presented here differs from Kaluza's in one essential point; we ascribe physical reality to the fifth dimension whereas in Kaluza's theory this fifth dimension was introduced only in order to obtain new components of the metric tensor representing the electromagnetic field. Kaluza assumes the dependence of the field variables on the four coördinates x^1, x^2, x^3, x^4 and not on the fifth coördinate x^0 when a suitable coördinate system is chosen.

It is clear that this is due to the fact that the physical continuum is, according to our experience a four dimensional one. We shall show, however, that it is possible to assign some meaning to the fifth coördinate without contradicting the four dimensional character of the physical continuum.

In the first chapter of our paper we present Kaluza's original theory; in the second, its new generalization. This is done in order to make the reading easier.

In the appendix we simplify the derivation of the field equations by generalizing the tensor calculus for the case of tensor densities.

I. THE KALUZA THEORY[1]

1. We consider a five dimensional space (4 + 1 dimensions) with the metric

$$d\sigma^2 = \gamma_{\mu\nu} dx^\mu dx^\nu. \tag{1}$$

2. We assume that the space is "cylindrical" with respect to a definite vector field. Analytically this means: There exists a contravariant vector A^ν so that if τ is an infinitesimally small constant, then

$$\delta x^\nu = \tau \cdot A^\nu \tag{2}$$

is an infinitesimal displacement of the space points. The element must not

[1] Contained in part in O. Klein, Zeitschrift f. Phys., **37** (1926), 895-906, A. Einstein, Sitzungsberichte d. Preuss. Akademie d. Wissenschaften, Physik.-mathemat. Klasse, 1927, 23-30.